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COMONOTONICITY, CORRELATION ORDER AND PREMIUM PRINCIPLES

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ABSTRACT

In this paper, we investigate the notion of dependency between risks and its effect on the related stop-loss premiums. The concept of comonotonicity, being an extreme case of dependency, is discussed in detail. For the bivariate case, it is shown that, given the distributions of the individual risks, comonotonicity leads to maximal stop-loss premiums. Some properties of stop-loss order preserving premium principles are considered. A simple proof is given for the sub-additivity property of Wang's premium principle.

Keywords: Dependency, correlation order, comonotonicity, stop-loss premium, premium principle.

1 Introduction

In many situations, individual risks are correlated since they are subject to the same claim generating mechanism or are influenced by the same economic/physical environment. For instance, the individual risks of an earthquake risk portfolio which are located in the same geographic area are correlated since individual claims are contingent on the the occurrence and severity of the same earthquake. As another example, consider a bond portfolio. Individual bond default experience may be conditionally independent for given market conditions. However, the underlying economic environment (e.g. interest rates) may affect all individual bonds in the market in a similar way.

In traditional risk theory, individual risks are usually assumed to be independent, mainly because the mathematics for correlated risks are less tractable. Consequently, the aggregate claims distribution and the stop-loss premiums are evaluated under the independence assumption.

Intuitively, with the presence of positive correlation, the law of large numbers will no longer hold and the aggregate risk may exhibit greater deviation than in the case of independent risks. Therefore, for a positively correlated risk portfolio, the independence assumption would probably under-estimate the stop-loss premiums.

In this paper, we will assume that each risk is a non-negative real-valued random variable defined on some fixed probability space. Each risk is assumed to have a finite mean. One main theme of this paper is to investigate the effect of correlation on stop-loss premiums when the assumption of mutually independence of the individual risks no longer holds.

A standard way of modeling situations where two individual risks X_i $(i = 1, 2)$ are subject to the same external mechanism is to use a secondary mixing distribution. The uncertainty about the external mechanism is then described by a structure parameter z, which is a realisation of a random variable Z. The aggregate claims can then be seen as a two-stage process: First, the external parameter $Z = z$ is drawn from the distribution function F_Z of Z; the claim amount of each individual risk X_i is then obtained as a realization from the conditional distribution function $F_{X_i}(x_i \mid Z = z)$ of X_i.

In this paper, we will introduce a special type of such a mixing model, namely the case where the conditional claim amounts $X_i \mid Z = z$ are degenerate and non-decreasing functions of z. Such a model is in a sense an extreme form of a mixing model, as in this case the external parameter $Z = z$ completely determines the aggregate claims. Risks that can be modeled by such a mixing model are said to be *comonotonic*.

In the following section, we will introduce some notations and definitions. In Section 3 the concept of comonotonicity and its close relation with Fréchet bounds for bivariate distribution functions is considered. In Section 4 the relation between comonotonicity and some of the results in Dhaene and Goovaerts (1996) is explored. In Section 5 we discuss the behavior of some premium principles in case that the risks involved are not mutually independent. Finally, in Section 6 a simple proof is given for the sub-additivity property of the class of premium principles introduced in Wang (1995, 1996).

2 Notations and Definitions

For a risk X (i.e. a non-negative real valued random variable), we denote its cumulative distribution function (cdf) and its decumulative distribution function (ddf) by F_X and S_X respectively:

$$F_X(x) = \Pr\{X \leq x\}, \quad 0 \leq x < \infty,$$

$$S_X(x) = \Pr\{X > x\}, \quad 0 \leq x < \infty.$$

The cdf F_X is non-decreasing while the ddf S_X is non-increasing. The relation between them is given by $S_X(x) = 1 - F_X(x)$.

In general, F_X is not one-to-one so that we have to be cautious in defining its inverse. The same remark holds for S_X. We will define F_X^{-1} and S_X^{-1} as follows:

$$\begin{aligned} F_X^{-1}(q) &= \inf\{x : F_X(x) \geq q\}, \qquad 0 < q < 1 \\ S_X^{-1}(q) &= \inf\{x : S_X(x) \leq q\}, \qquad 0 < q < 1. \end{aligned}$$

Remark that F_X^{-1} is non-decreasing, S_X^{-1} is non-increasing and $S_X^{-1}(q) = F_X^{-1}(1-q)$.

The traditional Monte Carlo simulation method is based on the following result.

Lemma 1 *For any risk X and any random variable U which is uniformly distributed on $(0,1)$, we have that X and $F_X^{-1}(U)$ have the same cdf.*

Now, consider two risks X and Y. We denote the bivariate cdf and the bivariate ddf of (X,Y) by $F_{X,Y}$ and $S_{X,Y}$ respectively. Hence,

$$F_{X,Y}(x,y) = \Pr\{X \leq x,\ Y \leq y\}, \quad 0 \leq x, y < \infty,$$

$$S_{X,Y}(x,y) = \Pr\{X > x,\ Y > y\}, \quad 0 \leq x, y < \infty.$$

From the bivariate cdf and ddf of (X,Y) we can find the marginal cdf's and ddf's of X and Y respectively by

$$F_{X,Y}(x,\infty) = F_X(x), \qquad F_{X,Y}(\infty,y) = F_Y(y),$$

$$S_{X,Y}(x,-\infty) = S_X(x), \qquad S_{X,Y}(-\infty,y) = S_Y(y).$$

Note that

$$S_{X,Y}(x,y) = 1 - F_X(x) - F_Y(y) + F_{X,Y}(x,y)$$

so that in general, $S_{X,Y}(x,y)$ will not be equal to $1 - F_{X,Y}(x,y)$.

If X and Y are independent, then $F_{X,Y}(x,y) = F_X(x) \cdot F_Y(y)$ and $S_{X,Y}(x,y) = S_X(x) \cdot S_Y(y)$.

For any risk X and any $d \geq 0$, we define $(X-d)_+ = \max(0, X-d)$. The net stop-loss premium with retention d is then defined by $\mathrm{E}(X-d)_+$. The stop-loss premium can be written in terms of the ddf:

$$\mathrm{E}(X-d)_+ = \int_d^\infty S_X(x)dx.$$

For $d = 0$, we find the following expression for $\mathrm{E}X$:

$$\mathrm{E}X = \int_0^\infty S_X(x)dx.$$

This means that $\mathrm{E}X$ is equal to the area below the curve of S_X.

The expectation of X can also be expressed in terms of the inverse ddf of X:

$$\mathrm{E}X = \int_0^1 S_X^{-1}(q)dq.$$

Intuitively, this means that $\mathrm{E}X$ is the area below the curve of S_X^{-1}. A graphical presentation of this result is given in Wang (1996).

3 Comonotonicity and Fréchet Bounds

The concept of *comonotonicity* was introduced by Schmeidler (1986) and Yaari (1987) and has since then played an important role in economic theories of decision under risk and uncertainty.

Definition 1 *Two risks X and Y are said to be Yaari-comonotonic if one of the following equivalent conditions holds:*

- *There is no pair of states of nature ω_1 and ω_2 such that $X(\omega_1) < X(\omega_2)$ and $Y(\omega_1) > Y(\omega_2)$.*
- *There exists a random variable Z and non-decreasing functions u and v on R such that $X = u(Z)$ and $Y = v(Z)$.*
- *There exist continuous non-decreasing functions u and v on R such that $u(z)+v(z) = z$ for all $z \epsilon R$ and $X = u(X+Y)$ and $Y = v(X+Y)$.*

The implications (3) $\Longrightarrow$ (2) $\Longrightarrow$ (1) are trivial. A proof for (1) $\Longrightarrow$ (3) can be found in Denneberg (1994). The term "comonotone" is an abbreviation for "common monotonic". Yaari-comonotonicity of two risks means that these risks are not able to compensate each other.

Yaari-comonotonicity is sensitive to random variables being changed on sets of probability zero. Roëll (1985) has adopted a weaker notion of comonotonicity, defined in terms of joint distributions, which is invariant under changes occuring on sets of probabilty zero.

Definition 2 *Two risks X and Y are said to be Roëll-comonotonic if their bivariate cdf satisfies*

$$F_{X,Y}(x,y) = \min\left(F_X(x), F_Y(y)\right) \qquad \textit{for all } x, y \geq 0.$$

Assume that X and Y are Roëll-comonotonic. From the definition above we see that, in order to find the probability of the outcomes of X and Y being less than x and y respectively, one simply takes the probability of the least likely of these two events.

It is easy to verify that a necessary and sufficient condition for Roëll-comonotonicity is also given by

$$S_{X,Y}(x,y) = \min\left(S_X(x), S_Y(y)\right) \qquad \text{for all } x, y \geq 0.$$

Theorem 2 *Yaari-comonotonicity implies Roëll-comonotonicity.*

Proof. Let X and Y be two Yaari-comonotonic risks. From Definition 1 we find that $X = u(Z)$ and $Y = v(Z)$ with u and v non-decreasing functions.
We have that

$$\begin{aligned} X &\leq x \Longleftrightarrow u(Z) \leq x \Longleftrightarrow Z \epsilon A \\ Y &\leq y \Longleftrightarrow v(Z) \leq y \Longleftrightarrow Z \epsilon B \end{aligned}$$

where A and B are intervals of the form $[0, d]$ or $[0, d[$.
As $A \subseteq B$ or $B \subseteq A$,we find

$$\begin{aligned} F_{X,Y}(x,y) &= \Pr\{Z\epsilon A, Z\epsilon B\} = \min(\Pr\{Z\epsilon A\}, \Pr\{Z\epsilon B\}) = \min(\Pr\{X \leq x\}, \Pr\{Y \leq y\}) \\ &= \min(F_X(x), F_Y(y)) \end{aligned}$$

which proves the theorem. ■

In the remainder of this paper, we will always assume Roëll-comonotonicity when we say that two risks are comonotonic.

The concept of comonotonicity is closely related to the following result, which is usually attributed to both Hoeffding (1940) and Fréchet (1951).

Theorem 3 *The joint cdf $F_{X,Y}(x,y)$ of the risks X and Y is constrainted from above and below by*

$$\max(F_X(x) + F_Y(y) - 1, 0) \leq F_{X,Y}(x,y) \leq \min(F_X(x),\ F_Y(y)).$$

Let $R(F_X, F_Y)$ be the class of all bivariate distributed random variables with marginals F_X and F_Y respectively. The bounds in Theorem 3 hold for all (X, Y) in $R(F_X, F_Y)$.
In order to show that the Fréchet bounds are reachable within the class of all risks with given marginal distributions, first remark that for any risk X and any $q\epsilon(0,1)$ and $x \geq 0$ we have that $F_X^{-1}(q) \leq x \Longleftrightarrow q \leq F_X(x)$. Now let U be any uniformly distributed random variable on $(0,1)$. Using the equivalence relation, it is easy to verify that $(F_X^{-1}(U), F_Y^{-1}(U))$ $\epsilon\ R(F_X, F_Y)$ and has a bivariate cdf given by the Fréchet upper bound:

$$F^u(x,y) = \min(F_X(x),\ F_Y(y)).$$

Similarly, we find that $(F_X^{-1}(U), F_Y^{-1}(1-U))\ \epsilon\ R(F_X, F_Y)$ and has a bivariate cdf given by

$$F^l(x,y) = \max(F_X(x) + F_Y(y) - 1, 0),$$

which corresponds to the Fréchet lower bound.

The concept of comonotonicity can be explained in terms of Monte Carlo simulation by inversion of uniform distributions. Assume that X and Y are comonotonic risks. We have that for U being a uniformly distributed random variable on $(0,1)$, $(F_X^{-1}(U), F_Y^{-1}(U))$

is comonotonic and has the same bivariate cdf as (X, Y). Hence, in order to simulate comonotonic risks, one needs to generate only one sample of random uniform numbers and insert in F_X^{-1} and F_Y^{-1} to get a sample of pairs of (X, Y).
By contrast, if X and Y are independent, then one needs to generate two independent samples of random uniform numbers and then insert these two sets in F_X^{-1} and F_Y^{-1}, respectively.

Recall that X and Y are (positively) perfectly correlated if and only if there exist real numbers $a > 0$ and b such that $Y = aX + b$, except, perhaps, for values of X with zero probability. It is easy to prove that positive perfect correlation of X and Y implies $F_{X,Y}(x, y) = \min\left(F_X(x),\ F_Y(y)\right)$. Hence, comonotonicity is an extension of the concept of positive perfect correlation.
Consider e.g.

$$X_1 = \begin{cases} X, & X \leq d \\ d, & X > d, \end{cases} \qquad X_2 = \begin{cases} 0, & X \leq d \\ X - d, & X > d. \end{cases}$$

Then X_1 can be interpreted as the part of total claims to be covered by the primary insurer and X_2 the part to be covered by the stop-loss reinsurer. It follows that X_1 and X_2 are *not* perfectly correlated since one cannot be written as a function of the other. However, since X_1 and X_2 are non-decreasing functions of the original risk X, they are comonotonic. This means that their bivariate cdf is given by

$$F_{X_1,X_2}(x, y) = \min\left(F_{X_1}(x),\ F_{X_2}(y)\right) = \begin{cases} F_X(x), & x \leq d \\ F_X(d+y), & x > d. \end{cases}$$

More generally, we can say that most risk sharing schemes (between insurer and reinsurer, or between insured and insurer) lead to partial risks that are comonotonic. The only restriction that has to hold is that both risk sharing partners have to bear more (or at least as much) if total claims increase.

The following theorem elucidates the importance of comonotonicity.

Theorem 4 *For comonotonic risks X and Y, we have*

$$\begin{aligned} F_{X+Y}^{-1} &= F_X^{-1} + F_Y^{-1} \\ S_{X+Y}^{-1} &= S_X^{-1} + S_Y^{-1}. \end{aligned}$$

Proof. A proof for the case of Yaari-comonotonic risks can be found in Denneberg (1994). If X and Y are Roëll-comonotonic, then the proof follows from the fact that (X, Y) has the same bivariate cdf as the Yaari-comonotonic pair $(F_X^{-1}(U), F_Y^{-1}(U))$ with U a uniformly distributed random variable on $(0, 1)$. ■
Theorem 4 states that comonotonicity implies additivity of the inverse (de-)cumulative distribution functions.

4 Stop-Loss Order and Correlation Order

In this section, we introduce some ordering relations between risks and between pairs of risks. We derive the extremal stop-loss premiums in the class of bivariate distributions with given marginals.
First, we introduce the stop-loss order.

Definition 3 *A risk X is said to precede a risk Y in stop-loss order, written $X \leq_{sl} Y$, if for all retentions $d \geq 0$, the net stop-loss premium for risk X is smaller than that for risk Y:*

$$\mathsf{E}(X-d)_+ \leq \mathsf{E}(Y-d)_+.$$

More details on this partial order between distribution functions can be found e.g. in Kaas *et al* (1994) and Müller (1996).

In Dhaene and Goovaerts (1996) ordering relations are investigated for the elements of the class $R(F_X, F_Y)$.

Definition 4 *Let (X_1, Y_1) and (X_2, Y_2) be two elements of $R(F_X, F_Y)$. We say that (X_1, Y_1) is less correlated than (X_2, Y_2), written $(X_1, Y_1) \leq_{corr} (X_2, Y_2)$, if either of the following equivalent conditions holds:*

- *For all non-decreasing functions f and g for which the covariances exist,*

$$\mathsf{Cov}(f(X_1), g(Y_1)) \leq \mathsf{Cov}(f(X_2), g(Y_2)).$$

- *For all $x, y \geq 0$, the following inequality holds:*

$$F_{X_1,Y_1}(x,y) \leq F_{X_2,Y_2}(x,y).$$

- *For all $x, y \geq 0$, the following inequality holds:*

$$S_{X_1,Y_1}(x,y) \leq S_{X_2,Y_2}(x,y).$$

The correlation order is a partial order between bivariate distributions in $R(F_X, F_Y)$ and expresses the idea that two random variables with given marginals are more "positively correlated" when they have some joint distribution than some other one.
The second (thirth) condition implies that the probability that both random variables of a couple realize small (large) values is smaller for the least correlated couple.

In many situations, insurance risks tend to act "more similarly" than in the independent case. A useful concept of bivariate dependency which can be used in such situations is positive quadrant dependency.

Definition 5 *The risks X and Y are said to be positively quadrant dependent, written $PQD(X,Y)$, if either of the following equivalent conditions holds:*

- *For all non-decreasing functions for which the covariances exist, we have that*

$$\mathsf{Cov}(f(X), g(Y)) \geq 0.$$

- *For all $x, y \geq 0$, the following inequality holds:*

$$F_{X,Y}(x,y) \geq F_X(x).F_Y(y).$$

- *For all $x, y \geq 0$, the following inequality holds:*

$$S_{X,Y}(x,y) \geq S_X(x).S_Y(y).$$

It follows immediately that $PQD(X,Y)$ is equivalent to saying that X and Y are more correlated (in the sense of Definition 4) than if they were independent.

From our earlier results and the Fréchet bounds we find that perfect positive correlation implies comonotonicity, which in turn implies positive quadrant dependency. Hence, for any risks X and Y we have:

$$X \text{ and } Y \text{ positive perfect correlated} \Longrightarrow X \text{ and } Y \text{ comonotone} \Longrightarrow \text{PQD}\,(X,Y)\,.$$

We can also introduce the notion of negative quadrant dependency, which is in a sense the opposite of positive quadrant dependency.

Definition 6 *The risks X and Y are said to be negatively quadrant dependent, written $NQD(X,Y)$, if either of the equivalent conditions in Definition 5 holds, with $\geq$ replaced by $\leq$.*

The notion of "negative quadrant dependency" can be used to describe situations where insurance risks are less correlated than in the independent case.

In the following lemma we derive expressions for stop-loss premiums of a sum in terms of the bivariate (de-)cumulative distribution function.

Lemma 5 *For all $d \geq 0$ we have*

$$\mathsf{E}(X+Y-d)_+ = \mathsf{E}(X) + \mathsf{E}(Y) - d + \int_0^d F_{X,Y}(x, d-x)dx$$

$$\mathsf{E}(X+Y-d)_+ = \mathsf{E}(X-d)_+ + \mathsf{E}(Y-d)_+ + \int_0^d S_{X,Y}(x, d-x)dx.$$

Proof. The first expression is established in Dhaene and Goovaerts (1996). Rewriting this expression we obtain

$$\begin{aligned} \mathsf{E}(X+Y-d)_+ &= \textstyle\int_0^\infty S_X(x)dx + \int_0^\infty S_Y(x)dx - d \\ &\quad + \textstyle\int_0^d [1 - S_X(x) - S_Y(1-x) + S_{X,Y}(x, d-x)]dx \\ &= \textstyle\int_d^\infty S_X(x)dx + \int_d^\infty S_Y(x)dx + \int_0^d S_{X,Y}(x, d-x)dx \end{aligned}$$

which is the second expression. ■

From Definition 4 and Lemma 5 we immediately find the following result (see Dhaene and Goovaerts, 1996).

Theorem 6 *Let* (X_1, Y_1) *and* (X_2, Y_2) *be elements of* $R(F_X, F_Y)$. *If*

$$(X_1, Y_1) \leq_{corr} (X_2, Y_2)$$

then

$$X_1 + Y_1 \leq_{sl} X_2 + Y_2.$$

Theorem 6 states that correlation order between two couples of random variables with given marginal distribution functions implies a stop-loss order between their respective sums.

Using Fréchet's result and the fact that $(S_X^{-1}(U), S_Y^{-1}(U))$ and $(S_X^{-1}(U), S_Y^{-1}(1-U))$ are both elements of $R(F_X, F_Y)$, we can prove the following theorem.

Theorem 7 *Let* U *be uniformly distributed on* $(0,1)$. *Then for any pair of risks* (X, Y) *the following ordering relations hold:*

- $(F_X^{-1}(U), F_Y^{-1}(1-U)) \leq_{corr} (X, Y) \leq_{corr} (F_X^{-1}(U), F_Y^{-1}(U))$
- $F_X^{-1}(U) + F_Y^{-1}(1-U) \leq_{sl} X + Y \leq_{sl} F_X^{-1}(U) + F_Y^{-1}(U)$.

From Theorem 7 we see that the Fréchet upper bound yields the maximum stop-loss premiums in the class of all bivariate distributions with given marginals:

$$\mathsf{E}(X+Y-d)_+ \leq \int_0^1 [F_X^{-1}(q) + F_Y^{-1}(q) - d]_+ dq.$$

For an expression of this upper bound in case of exponential marginals, see Heilmann (1986). Similarly, the Fréchet lower bound yields the minimal stop-loss premiums in the class of all bivariate distributions with given marginals:

$$\mathsf{E}(X+Y-d)_+ \geq \int_0^1 [F_X^{-1}(q) + F_Y^{-1}(1-q) - d]_+ dq.$$

Hence, for any pair of risks (X, Y), we have found an upper bound and a lower bound for the stop-loss premiums of $X+Y$. These bounds are expressed in terms of the (inverse) cdf's of X and Y. Remark that these bounds hold for any pair of random variables contained in $R(F_X, F_Y)$, regardless of their dependency structure.
In the limiting case of full reinsurance, i.e. $d=0$, the lower and the upper bound both reduce to $\int_0^1 [F_X^{-1}(q) + F_Y^{-1}(q)]dq$ which is equal to $\mathrm{E}X+\mathrm{E}Y$.

5 Premium Principles

The net premium of a risk X is defined as the expectation of X. Insurers usually charge a risk-adjusted premium, being the sum of the net premium and some risk load. A premium principle is a rule $\pi : X \mapsto [0, \infty)$ that assigns a positive value (the risk-adjusted premium) to any risk X. We will assume that risks with the same cdf lead to the same risk-adjusted premium.
A desirable property for a premium principle is that it preserves stop-loss order, i.e. $X \leq_{sl} Y$ implies that $\pi(X) \leq \pi(Y)$ (see e.g. Kaas et al, 1994).
From Theorem 6 we immediately find the following result:

Theorem 8 *Let $\pi : X \mapsto [0, \infty)$ be a premium principle which preserves stop-loss order, and (X_1, Y_1) and (X_2, Y_2) be elements of $R(F_X, F_Y)$. If*

$$(X_1, Y_1) \leq_{corr} (X_2, Y_2)$$

then

$$\pi\left(X_1 + Y_1\right) \leq \pi\left(X_2 + Y_2\right).$$

From the Theorems 7 and 8, we find the following corollary.

Corollary 9 *Let π be a premium principle which preserves stop-loss order. Then we have*

$$\pi\left(F_X^{-1}(U) + F_Y^{-1}(1-U)\right) \leq \pi(X+Y) \leq \pi\left(F_X^{-1}(U) + F_Y^{-1}(U)\right).$$

The corollary states that the risk-adjusted premium of a sum of two risks is maximal if the two risks are comonotonic. As comonotonic risks can be considered as bets on the same event, neither of them is a hedge against the other. So it seems a desirable property that the premium of the sum is maximal in this case.
On the other side, we see that the premium of $X+Y$ is minimal if (X, Y) has the same bivariate distribution as $\left(F_X^{-1}(U), F_Y^{-1}(1-U)\right)$. In this case the combination of both risks

leads to an optimal hedge as the higher the one risk, the lower the other one will be. So it seems to be a desirable property that the lowest risk-adjusted premium is obtained in this case.

A premium principle is called additive within a given class of risks if the premium for the sum of any two risks taken from this class equals the sum of the individual premiums. A premium principle is said to be sub-additive (super-additive) if the premium for the sum is not larger (not smaller) than the sum of the individual premiums.

Corollary 10 *If a premium principle preserves stop-loss order and is additive for independent risks, then it is sub-additive for negative quadrant dependent risk, and super-additive for positive quadrant dependent risks:*

$$\pi(X+Y) \leq \pi(X)+\pi(Y) \quad \text{if } NQD(X,Y)$$

$$\pi(X+Y) \geq \pi(X)+\pi(Y) \quad \text{if } PQD(X,Y).$$

As a special case of Corollary 10, we find that a stop-loss order preserving premium principle which is additive for independent risks, is super-additive for comonotonic risks. Remark that the well-known exponential premium principle satisfies the conditions of Corollary 10, see e.g. Kaas et al. (1994).

In the following corollary, we consider the case that the premium principle is additive for comonotonic risks.

Corollary 11 *If a premium principle preserves the stop-loss order and is additive for comonotonic risks, then it is sub-additive:*

$$\pi(X+Y) \leq \pi(X)+\pi(Y) \quad \text{for all risks } X \text{ and } Y.$$

Hence, premium principles which satisfy the conditions of Corollary 11 always give a volume discount.

The question which kind of conditions (the one from Corollary 10 or 11) are preferable, depends upon the situation under consideration.

Consider two pairs of risks (X,Y) and (X',Y') with the same marginal cdf's. Assume that X and Y are comonotonic parts of one combined risk $X+Y$. The premium for this combined risk equals $\pi(X+Y)$. Further, assume that X' and Y' are the risks belonging to two different policyholders. These policyholders pay a total premium equal to $\pi(X)+\pi(Y)$ which is independent of the correlation structure between the two risks. The insurer will prefer the risks X' and Y' over X and Y because comonotonic risks are bets on the same

event. The insurer can incorporate this preference in his premium structure by choosing a premium principle that is super-additive for such risks, i.e. $\pi(X+Y) \geq \pi(X)+\pi(Y)$ for all X and Y that are comonotonic. This means that the insurer is not willing to give a reduction in the risk load for a combined policy of comonotonic risks.
Next, assume that each policyholder is free to split up his risk and buy separate policies for this splitted risks from the same insurer. In this situation , the insurer should use a sub-additive premium principle, i.e. $\pi(X+Y) \leq \pi(X)+\pi(Y)$ for all risks X and Y, since otherwise the policyholder will be better off by buying separate policies.
We can conclude that if the insurer is not willing to give a reduction for a combined policy of comonotone risks and if he wants to avoid splitting up of risks (assumed that it is possible), then he should use a premium principle that is additive for comonotonic risks.
Remark that if splitting up of risks is not possible, then the insurer can use a premium principle that is super-additive for comonotonic risks. An example is catastrophe insurance, where the insurer will only be willing to insure a larger part of the complete risk at a higher risk load.

Wang (1996) proposes to compute the risk-adjusted premiums by the following premium principle:

$$H_g[X] = \int_0^\infty g[S_X(x)]dx = \int_0^1 S_X^{-1}(q)dg(q),$$

where g is a non-decreasing concave function with $g(0) = 0$ and $g(1) = 1$.
Note that for $g(x) = x$ $(0 \leq x \leq 1)$, $H_g[X] = \mathsf{E}X$.

The interpretation of this class of premium principles is clear: First, the original ddf of the risk X is replaced by a new ddf $g(S_X)$ which gives more weight to the right-tail.
Then the risk adjusted premium is computed as the expectation of X under the new ddf. Wang's premium principle preserves some common ordering of risks such as stop-loss ordering.

Theorem 12 *Wang's premium principle preserves stop-loss order, i.e.*

$$X \leq_{sl} Y \Longrightarrow H_g[X] \leq H_g[Y].$$

Moreover, it is additive in the class of comonotonic risks,

$$H_g[X+Y] = H_g[X] + H_g[Y] \quad \textit{for comonotonic risks } X \textit{ and } Y.$$

Proof. See Wang (1996). ■

It can be shown that transforming the ddf as is done in Wang's premium principle, is the only way to get comonotonic-additive and stop-loss order preserving premium principles. Hence, outside Wang's class of premium principles there are no premium principles that have these two properties simultaneously, see Denneberg (1994).

6 A Simple Proof of Sub-additivity

In Wang (1995), a proof (due to Ole Hesselager) is given for the sub-additivity property of Wang's premium principle in the special case where $g(x)$ is of the form $g(x) = x^c$. It was stated in Wang (1996) that the proof for this special case could readily be generalized to other functions g. However, Denneberg pointed out (via personal communication) that this statement is not true. As an application of the present paper, now we can give a correct and simple proof for the sub-additivity theorem.

Theorem 13 *For any two risks, regardless of their dependency relation, we have that*

$$H_g[X+Y] \leq H_g[X] + H_g[Y].$$

Proof. The proof follows immediately from Corollary 11 and Theorem 12. ■

Remark that the upper bound in Theorem 13 corresponds to the case that both risks are comonotonic.

From Corollary 9 and Theorem 12 we also find the following lower bound for $H_g[X+Y]$:

$$H_g[X+Y] \geq H_g[S_X^{-1}(U) + S_Y^{-1}(1-U)].$$

The lower bound corresponds to the case that both risks are maximum hedges against each other.

Finally, remark that if X and Y are positive quadrant dependent, then a better lower bound is given by $H_g\left[X^{ind} + Y^{ind}\right]$ where X^{ind} and Y^{ind} are mutually independent and have the same marginal cdf's as X and Y respectively.

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